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S. K. Turitsyn, A. M. Rubenchik, M. P. Fedoruk

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On the Theory of the Modulation Instability in Optical Fibre Amplifiers

Sergei K. Turitsyn,[1,*] Alexander M. Rubenchik,[2] and Michail P. Fedoruk[3]

[1] *Photonics Research Group, Aston University, Birmingham, B4 7ET, UK*

[2] *Lawrence Livermore National Laboratory, Livermore, CA 94550, USA*

[3] *Institute of Computational Technologies, Novosibirsk 630090 Russia*

[*] *Corresponding author: s.k.turitsyn@aston.ac.uk*

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Abstract: The modulation instability (MI) in optical fibre amplifiers and lasers with anomalous dispersion leads to CW radiation break-up and growth of multiple pulses. This can be both a detrimental effect limiting the performance of amplifiers, and also an underlying physical mechanism in the operation of MI-based devices. Here we revisit the analytical theory of MI in fibre optical amplifiers. The results of the exact theory are compared with the previously used adiabatic approximation model and the range of applicability of the later is determined.

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Modulation instability (MI) is a fundamental nonlinear effect [1-3] that manifests itself in optics e.g. as a spontaneous break-up of a continuous wave (CW) radiation with high enough power into modulated light wave or periodic train of pulses (it is not possible to overview all the literature on MI effects in optics, however here we focus only on MI in active optical media see e.g. [1-15] and references therein). In optical fibre MI occurs as a result of interplay between the effects of the anomalous group-velocity dispersion (GVD) and self-phase modulation. In active fibre, the modulation instability is enhanced by interactions with optical noise providing seeding perturbations over a range of wavelengths [3]. The MI effect might play a destructive role – leading to degradation of a quality of optical waves and beams. For instance, amplification of powerful laser radiation in optical fibre amplifiers with anomalous dispersion might suffer from the MI effect [3] that leads to the break-up of CW radiation, exponential growth of modulations and, as a result, the appearance of multiple pulses and irregularities in the power distribution. The problem, in particular, might be relevant to Erbium-doped optical amplifiers widely used at the telecommunication window near 1.5 micron and having anomalous dispersion in this spectral range. However, MI can also be exploited in a constructive way, for instance, as a technique to generate an optical pulse train or as a passive mode-locking mechanism in fibre lasers [4-10]. In this context, MI is a passive nonlinear effect that has cost advantage over schemes using ultrafast modulators. An important feature of this technique is that the generation of continuous streams of short-pulses via MI can be realised at high repetition rates. As a nonlinear fibre effect sensitive to dispersion, MI is also very attractive for various measurement techniques [11, 12]. Recent progress in micro-structured optical fibres offers new opportunities for the control of dispersive properties and, thus, to new potential applications of MI across a broad spectral range. Quantitative analysis of the modulation instability is important for design and optimisation of fibre lasers and amplifiers in which the wave intensity grows up exponentially and MI dramatically intensifies nonlinear instabilities. As we show below, despite a number of publications, some important aspects of the instability development over finite device distance have not yet been comprehensively studied. In this work we revisit the theory of MI in active fibre and compare exact analytical results to the adiabatic approximation approach [3].

Over a wide range of physical parameters, propagation of the optical field down a fibre amplifier at leading order is described by the nonlinear Schrödinger equation (NLSE) with the gain terms (also called Ginzburg-Landau equation):

$$i\frac{\partial \Psi}{\partial z}-\frac{\beta_2}{2}\Psi_{tt}+\gamma\,|\,\Psi\,|^2\ \Psi=i\frac{g_0}{2}\Psi+i\,\frac{g_0T_2^2}{2}\Psi_{tt} \qquad (1)$$

Here β_2 is the group velocity dispersion; nonlinear parameter $\gamma = 2\pi\, n_2\,/(\lambda_0 A_{eff})$ (λ_0 is the operational wavelength, n_2 nonlinear refractive index, A_{eff}- effective area of the fibre); g_0 is the small signal gain of the amplifier. The parameter T_2 characterizes the gain bandwidth of an amplifier (or effect of external filtering). An optical field propagates here from z = 0 to z = L. The instability in the amplifier is similar to the problem of propagation in non-uniform media [13]. Consider the modulation instability of the CW field: $\Psi(z,t)=(\sqrt{P_0}+a+ib)\times\exp[\frac{g_0 z}{2}+iP_0\int\gamma(z')dz']$, here $\gamma(z)=\gamma(0)\exp[g_0\, z]$.

Perturbation to the power evolution then can be found as: $|\,\Psi(z,t)\,|^2=(P_0+2a(z,t)\sqrt{P_0}+a^2+b^2)\times\exp[g_0 z]$. Assuming $a,b<<\sqrt{P_0}$ and expressing the fields a,b through the corresponding Fourier modes $a_\omega, b_\omega \propto \exp[-i\omega t]$ (for simplicity of notations, we omit in what follows the index ω) yields the standard linear evolution equations (2) for the spectral modes of perturbations with the initial conditions to the Caushy problem $a(0), b(0)$. When $\gamma = const, T_2 = 0$, $a\propto\exp[ik_z z]$ leads to the standard MI relation [1]: $k_z^2=\frac{\beta_2\omega^2}{2}[\frac{\beta_2\omega^2}{2}+2\gamma P_0]$ with k_z increasing for small values of ω, reaching its maximum at $\omega_{\max}^2=-2\gamma P_0/\beta_2$, and approaching zero at $\omega_0^2=-4\gamma P_0/\beta_2$. In amplifiers, however, where the field power grows as $P(z)=P_0e^{g_0 z}$, the most unstable frequency of perturbation increases during the propagation due to the power exponential growth. To estimate the growth due to MI in an amplifying medium one can use the expression for the uniform MI, but replace constant power with the growing one $P_0 \to P_0 e^{g_0 z}$. This corresponds to the so-called adiabatic approximation (see e.g. [3]) in which it is assumed that the perturbation growth follows the intensity adiabatically and the standard NLSE expression with a z-dependent intensity $\gamma(z)$ can be used. In the inhomogeneous medium the instability evolution is described by the equation (2) that can be solved analytically. Introducing $\eta = \frac{2\omega\sqrt{-\beta_2 \gamma_0 P_0}}{g_0}$, $2s = g_0 T_2^2 \omega^2$, $\mu = \frac{-\beta_2 \omega^2}{g_0}$, $p^2 = \frac{\mu^2}{\eta^2} = \frac{-\beta_2 \omega^2}{4\gamma_0 P_0}$, the equations for a(z) and b(z) can be presented in the form:

$$\begin{aligned} \frac{da}{dz} &= \frac{g_0 \mu}{2} b(z) - s\, a(z), \\ \frac{db}{dz} &= \frac{\eta^2 g_0}{2\mu}(-p^2 + \exp[g_0 z])a(z) - s\, b(z). \end{aligned} \tag{2}$$

The solution to (2) can be presented through the Bessel functions $I_{i\mu}(x)$ and $K_{i\mu}(x)$ (compare to approaches used in [14] in context of short-scale self-focusing and in [15] for analysis of modulation instability in lossy fibres):

$$\begin{aligned} a(z) &= A\, I_{i\mu}(\eta\, e^{g_0 z/2}) + B\, K_{i\mu}(\eta\, e^{g_0 z/2}), \\ b(z) &= -\frac{\eta\, e^{g_0 z/2}}{2\mu}\{C\,(I_{i\mu-1} + I_{-i\mu+1}) + D\ (K_{i\mu-1} + K_{i\mu+1})\}. \end{aligned} \tag{3}$$

Here $A = [-a(0)\eta\, K'_{i\mu}(\eta) + b(0)\,\mu\, K_{i\mu}(\eta)]e^{-sz}$, $B = [a(0)\eta\, I'_{i\mu}(\eta) - b(0)\,\mu\, I_{i\mu}(\eta)]e^{-sz}$, $C = -\eta\mu^{-1} e^{g_0 z/2} A$, $D = \eta\mu^{-1} e^{g_0 z/2} B$. The solutions (3) are functions of three dimensionless parameters: $g_0 z, \mu, \eta$. The Stürmian theory [16] guarantees for the Sturm–Liouville problem (2) that the solutions (3) are growing with z under condition $\mu < \eta\, e^{g_0 z/2}$. For $\eta\, e^{g_0 z/2} > \mu$ and $\mu >> 1$ the leading term in the expansion of the exact solution reads:

$$I_{i\mu}(\eta\, e^{g_0 z/2}) \approx \frac{\exp[\sqrt{\eta^2 e^{g_0 z} - \mu^2} + \mu \arcsin(\frac{\mu\, e^{-g_0 z/2}}{\eta})]}{\sqrt{2\pi}\sqrt[4]{(\eta^2 e^{g_0 z} - \mu^2)}} + \ldots \tag{4}$$

$$K_{i\mu}(\eta\, e^{g_0 z/2}) \approx \frac{\sqrt{\pi}\times\exp[-\sqrt{\eta^2\, e^{g_0 z}-\mu^2}-\mu\arcsin(\frac{\mu\, e^{-g_0 z/2}}{\eta})]}{\sqrt[4]{4(\eta^2\, e^{g_0 z}-\mu^2)}}+...$$

It is seen that in this limit $K_{i\mu}$ is decaying and $I_{i\mu}$ is growing and the growth of perturbations in the amplifier is *super-exponential*. In the opposite limit $\mu > \eta\, e^{g_0 z/2}$. both $K_{i\mu}$ and $I_{i\mu}$ are oscillating. Note that the asymptotic behaviour of $I_{i\mu}$ not only justifies the use of the adiabatic approximation [3] in the limit $\mu >> 1$, but also provides the pre-exponential factor. The increment of growth in the adiabatic approximation is $\Gamma = 2f(\mu,\eta,g_0,L) - 2sL$, with $f(\mu,\eta,g_0,L)$ defined as:

$$f = (\eta^2 e^{g_0 L} - \mu^2)^{1/2} - \sqrt{\eta^2-\mu^2} + \mu\arcsin(\frac{\mu\, e^{-g_0 z/2}}{\eta}) - \mu\arcsin(\frac{\mu}{\eta});\ \frac{\mu}{\eta} < 1,$$

$$f = (\eta^2 e^{g_0 L} - \mu^2)^{1/2} + \mu\arcsin(\frac{\mu\, e^{-g_0 L/2}}{\eta});\ \frac{\mu}{\eta} > 1.$$

This estimate in many cases describes rather well the asymptotic growth (though without the pre-exponent term as in (4)), however, justification and limitations of this approach are not clear *a priori*. The important result of our work is that it gives a direct analytical expression for the dynamics of the perturbations for *any arbitrary initial fluctuations and any propagation distance*. The dimensionless scaling allows one to apply our analytical results to a range of physical problems. The power growth of the initial perturbations can be characterized by the increment factor (similar to the homogeneous case making comparison more convenient) defined as:

$\Gamma = 2\ln\big[a(L)/a(0)\big],\ \mu < \eta;\ \ \Gamma = 2\times\ln[a(z)/a(z^*)],\quad \mu > \eta,\quad z^* = \frac{1}{g_0}\ln[\frac{\mu^2}{\eta^2}].$ Here, we assume $a(0), a(z^*) \neq 0$. For large $a(L)/a(0)$ the increment is practically independent of boundary conditions. It should be stressed, however, that in the exact solutions (4) there are both growing and decaying solutions. For short propagation distances, both can contribute to the development of instability – a fact that is often overlooked considering MI. This means, in particular, that for short devices where MI does not have enough time/distance to develop into an asymptotic state with the growing mode dominating completely, the initial phase perturbations given by $b(0)$ might affect the growth increment of developing modulations. Initial conditions also become important near the cut-off of instability as the growth is not large near such points and it is influenced by the initial field perturbations. This is illustrated by Fig. 1 where the relative impact of the initial phase $b(0)$ and amplitude a(0) perturbations on the growing solution are shown (the coefficient before the growing solution $A=[-a(0)\eta K_{i\mu}^{'}(\eta)+b(0)\mu K_{i\mu}(\eta)]$). Here $a^2(0)+b^2(0)=1$ and $\phi=\tan^{-1}[b(0)/a(0)]$.

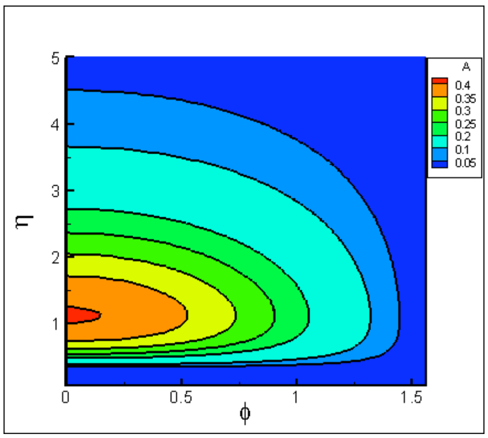

Figure 1. Counterplot of the coefficient $A=[-a(0)\eta K_{i\mu}^{'}(\eta)+b(0)\mu K_{i\mu}(\eta)]$ before the growing solution in the plane (η,ϕ) with $\mu=1$.

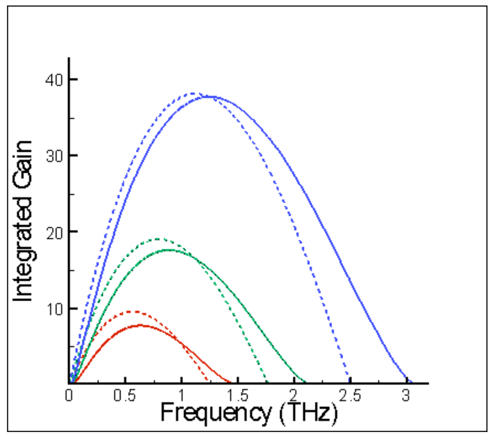

Figure 2. Gain $\Gamma(\nu)$ for $L=100\,m$, $G=30\,dB$ here $P_0=|\Psi_0|^2$: 50 (red), 100 (green) and 200 (blue) mW; solid lines – exact solutions, dashed - adiabatic approximation used in [3].

In general, the increment factor $\Gamma(\omega,\mu,p,s,G,L)$ is a multi-parametric function of the parameters $\omega,\mu,\ p,\ s,G$ and L, or in the real-world units $\omega,P_0,\beta_2,\gamma,T_2,G$ and L. Therefore, the existence of the analytical solution is very useful for design analysis. For fixed values of other parameters we have to determine the maximum value of the increment growth Γ as a function of ω. In a uniform media (g_0L=0), the most unstable mode corresponds to ω^2=1/2 and cut-off at ω^2=1. In contrast, in an amplifier, the most unstable value of ω increases during the propagation. For illustration we use here similar parameters as in Ref.

[3]: $\beta_2 = -20\ ps^2/km; \gamma = 10\ W^{-1}km^{-1}$; amplifier length $L = 100\ m$, the total gain $G = g_0\ L = 30\ dB$. Figure 2 shows the integrated gain $\Gamma(\nu)$ for several values of the input power. It is seen that the adiabatic approximation (dashed lines) being close to the exact solutions (solid lines) still deviates in determination of the frequency of the maximal instability. This might be critically important for design of MI-based lasers.

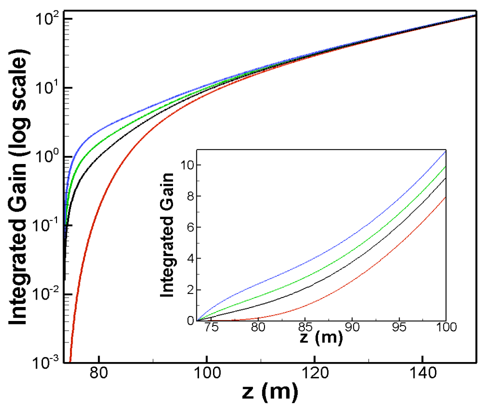

Figure 3: Integrated gain Γ (log scale) vs propagation distance for $a(z_*) = 1$ and $b(z_*) = 0$ (red line), $b(z_*) = 1$ (green line), $b(z_*) = 2$ (blue line); black line – $I_{i\mu}$, $P_0 = 50\ mW$. Inset – normal scale.

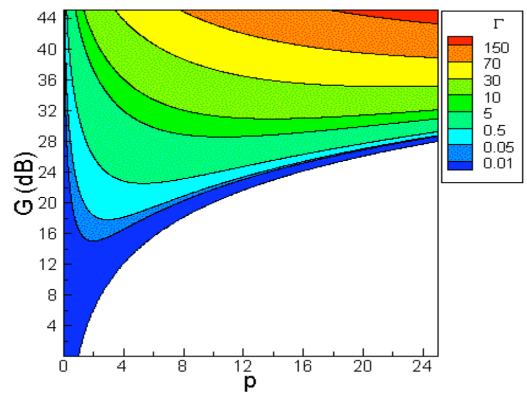

Figure 4: Counterplot Γ in the plane (p, G), $\eta^2/\mu = 0.03$. White zone corresponds to the oscillating solutions. The border between stable and unstable regimes is given by the condition: $G > \mu^2/\eta^2$.

Figure 3 illustrates the impact of the initial conditions on the instability growth (typically overlooked in studies limited by the analysis of the growth increment only) showing growth of the integrated gain Γ with distance for $a(z_*) = 1$ and different $b(z_*)$: $b(z_*) = 0$ – red line, $b(z_*) = 1$ – green line, $b(z_*) = 2$ – blue line. Here black line corresponds to $I_{i\mu}$. Figure 4 depicts the integrated gain x as a function of the normalized frequency p and the total gain G. Note that the ω corresponding to maximum MI growth shifts up with g_0L increasing, the cut-off takes place at $\omega > 1$ and the most unstable modes corresponds to $\omega > 1$. It means that the most unstable modes initially were stable and start to grow only later downstream. The effect of this sliding of the most unstable frequency with the development of MI in the optical fibre amplifier has direct impact on the operation MI-based fibre laser and generation of pulse trains using MI. For instance, in fibre lasers where MI triggers passive mode-locking the instability frequency should be in resonance with the resonator frequency and this sliding of the maximum of instability should be taken into account.

We have revisited the theory of modulation instability in fibre amplifiers. We found the complete analytical solutions of the linear growth that allows us to find the most unstable mode and calculated the power growth exactly - without restricting the consideration to the asymptotically growing mode as in most previous works. We demonstrated that for practical situations the growth of the perturbation is sensitive to the initial perturbation and to their phases. In many applications the initial perturbation fields are different from the plane wave and amplify from some other distribution than noise. Our results indicate how to modulate the signal to accelerate the breaking into shorter pulses and to optimise the design of the soliton laser. Our results are directly relevant to the modulation instability in optical fibre amplifiers and lasers, but the derived theory is rather general and can be applied in a variety of physical applications beyond fibre optics.

References

1. V. E. Zakharov and L. A. Ostrovsky, “Modulation instability: the beginning,” Physica D **238**, 540–548 (2009).
2. V.I.Bespalov and V.I.Talanov, “Filamentary Structure of Light Beams in Nonlinear Media," JETP Lett., **3**, 307–310 (1966).
3. G. P. Agrawal, “Modulation instability in erbium-doped fibre amplifiers,” IEEE PTL, **4(6)**, 562–564 (1992).

4. K. Tai, A. Tomita, J. L. Jewell, and A. Hasegawa, "Generation of subpicosecond solitonlike optical pulses at 0.3 THz repetition rate by induced modulational instability," Appl. Phys. Lett. **49**, 236–238 (1986).
5. P. V. Mamyshev, S. V. Chernikov, E. M. Dianov, and A. M. Prokhorov, "Generation of a high-repetition-rate train of practically noninteracting solitons by using the induced modulational instability and Raman self-scattering effects," Opt. Lett. **15**, 1365–1367 (1990).
6. S.V. Chernikov, D.J. Richardson, R.I. Laming, E.M. Dianov and D.N. Payne, “70 Gbit/s fiber based source of fundamental solitons at 1550nm,” Electron. Lett. **28**, 1210–1212 (1992).
7. M. Nakazawa, K. Suzuki, H. A Haus, “The modulational instability laser—Part I: Experiment,” IEEE J. Quantum Electron. **25,** 2036–2044 (1989).
8. P. Franco, F. Fontana, I. Cristiani, M. Midrio and M. Romagnoli, “Self-induced modulational-instability laser,” Opt. Lett **20**, 2009–2011 (1995).
9. C. J. S. de Matos, D. A. Chestnut, and J. R. Taylor, “Low-threshold self-induced modulational instability ring laser in highly nonlinear fiber yielding a continuous-wave 262-GHz soliton train,” Opt. Lett. **27**, 915–917 (2002).
10. Y. Gong, P. Shum, D. Tang, C. Lu, and X. Guo, “660GHz Solitons Source Based on Modulation Instability in Short Cavity,” Opt. Express **11**, 2480–2485 (2003).

11. M. Artiglia, E. Ciaramella, and B. Sordo, "Using modulation instability to determine Kerr coefficient in optical fibres," Electron. Lett., **31(12)**, p.1012–1013 (1995).

12. J. Fatome, S. Pitois, G. Millot, "Measurement of Nonlinear and Chromatic Dispersion Parameters of Optical Fibers using Modulation Instability," Optical Fiber Technology, **12(3),** 243–250. July (2006).

13. A. M. Rubenchik, S. K. Turitsyn, and M. P. Fedoruk, "Modulation instability in high power laser amplifiers," Opt. Express **18**, 1380–1388 (2010).

14. N. N. Rozanov and V. A. Smirnov, "Short-scale self-focusing of laser radiation in amplifying systems", Sov. J. Quantum Electron., **10(2)**, 232–237 (1980).

15. M. Karlsson "Modulational instability in lossy optical fibres", J. Opt. Soc. Am B, **12**, 2071 (1995).

16. E. L. Ince, *Ordinary Differential Equations* (Dover, 1956).